geographical analysis

Geographical Analysis ISSN 0016-7363

Minority–Majority Relations in the Schelling Model of Residential Dynamics

Itzhak Benenson[1], Erez Hatna[2]

[1]Department of Geography and Human Environment, Tel Aviv University, Tel Aviv, Israel, [2]Institute for Management Research, Radboud University, Nijmegen, The Netherlands

The Schelling model describing segregation between two groups of residential agents, reflects the most abstract, basic view of noneconomic forces motivating residential migrations: be close to people of ''your own'' kind. The model assumes that residential agents, located in neighborhoods where the fraction of ''friends'' is less than a predefined threshold value F, try to relocate to neighborhoods where this fraction is F or higher. For groups of equal size, Schelling's residential pattern converges either to complete integration (random pattern) or segregation, depending on F. We investigate Schelling model pattern dynamics as a function of F in addition to two other parameters—the ratio of groups' numbers, and neighborhood size. We demonstrate that the traditional integration–segregation pattern dichotomy should be extended. In the case of groups of different sizes, a wide interval of F-values exists that entails a third persistent residential pattern, one in which a portion of the majority population segregates while the rest remains integrated with the minority. We also demonstrate that Schelling model dynamics essentially depend on the formalization of urban agents' residential behavior. To obtain realistic results, the agents should be satisficers, and the fraction of the agents relocating irrespective of the neighborhood's state should be nonzero. We discuss the relationship between our results and real-world residential dynamics.

Introduction

The Schelling model's basic framework

The Schelling model of segregation (Schelling 1971, 1978) reflects a very basic, abstract view of noneconomic forces motivating residential migration: be close to people of ''your own'' kind. Formally, *residential agents* occupy cells of rectangular *residential space*, not more than one agent per cell. Agents are of two types, which we denote here as B(lue) and G(reen). Agents ''know'' the state of every cell in the neighborhood surrounding their own locations—whether a cell is occupied and, if so, by what type of resident.

Correspondence: Itzhak Benenson, Department of Geography and Human Environment, Tel Aviv University, Tel Aviv 69978, Israel
e-mails: bennya@post.tau.ac.il, erezh51@gmail.com

Submitted: December 8, 2008. Revised version accepted: August 24, 2010.

The model's basic assumption is as follows: an agent, located in the center of a neighborhood where the fraction of friends f is less than a predefined threshold value F (i.e., $f < F$) will want to relocate to a neighborhood where $f \geq F$ (Schelling 1978, p. 148).[1]

Schelling (1978, p. 138) used the model to examine ''some of the individual incentives and individual perceptions of difference that can lead collectively to segregation.'' To do so, he placed equal numbers of dimes and pennies on squares of a chessboard-sized sheet of paper and assumed that every dime or penny wanted ''something more than one-third'' of its neighbors within its 3×3 neighborhood to be like itself. Any dime or penny whose neighborhood did not meet this condition moved to the nearest square that did satisfy it (Schelling 1978, pp. 147–48).

Regardless of the initial pattern and the order of moves, the dimes and pennies in Schelling's experiment quickly segregated, and the pattern stalled. In this and similar experiments, Schelling demonstrated that for F above one-third, all patterns converge to a segregated equilibrium. He thus concluded that micro-behavior aimed at satisfying relatively weak individual demands (i.e., to reside within a neighborhood where $F \geq 1/3$ only), results in a completely segregated macropattern. He also noted that the properties of the equilibrium pattern depend on the value of F and that F can differ for dimes and for pennies (Schelling 1978, pp. 153–54), although he did not investigate these issues.

In this article, we claim that the traditional, integration–segregation description of the persistent patterns produced by the Schelling model should be extended. For the case of groups of unequal size, we demonstrate the existence of a wide interval of F-values that encompasses a third persistent residential pattern, one in which a portion of the majority group segregates while the rest remains integrated with the minority.

Our study aims to extend Schelling's theoretical framework. We thus ignore numerous real-world factors that can influence real-world residential choice, such as the hierarchical structure of residential space (e.g., apartment, entrance, building, and neighborhood), commuting and relocation costs, and distance from work.

Schelling model studies: state of the art

Standard settings and the basic dichotomy of persistent patterns

The settings that Schelling (1978) establishes can be described as follows:

S1. The city is a grid of cells, and each cell's neighborhood is a 3×3 square, truncated by the city's boundaries if they lie in close proximity.

S2. The city is populated by agents, each belonging to one of two groups of equal size, B(lue) and G(reen); $B : G = 1 : 1$.

S3. The initial distribution of agents on the grid city is random; a fraction of cells is vacant, with this number being sufficient to allow relocation.

S4. Model dynamics are considered in discrete time. At each time step, every agent estimates the fraction f of neighbors of its own type (friends) within the neighborhood (empty places are ignored).

S5. There exists a threshold value F, common to all agents, that represents the fraction of friends desired within each neighborhood. An agent located at the center of a neighborhood where $f < F$ decides to leave its cell and relocate to the *closest* empty cell satisfying the condition $f \geq F$.

S6. Information about a cell left by an agent becomes immediately available to all agents.

In this article, we call S1–S6 the Schelling model's standard settings.

According to Schelling (1971), for standard settings and $F < 1/3$, the model's residential pattern eventually converges to a pattern that is visually indistinguishable from the random one, while for $F \geq 1/3$, the model's residential pattern converges to a segregated pattern, characterized by one or more large (i.e., several times larger than the 3×3 neighborhood) homogeneous patches of B- and G-agents. These patches are separated by unpopulated boundaries (Fig. 1).

But the standard settings are incomplete. For example, they do not consider the important situation where an agent decides to relocate, but no vacancies are available satisfying the condition $f \geq F$. This feature implies that one should be cautious when comparing the results of different studies. The consequences of formal implementation of the same conditions may be only apparently equivalent yet substantively dissimilar.

Studies of the Schelling model

The Schelling model is studied mainly through simulation.[2] Research tends to focus on settings that are more general than the standard setting (S1–S6). Flache and Hegselmann (2001) investigated the model for irregular partitions of the plane and demonstrate that convergence toward either the segregated or the random pattern is

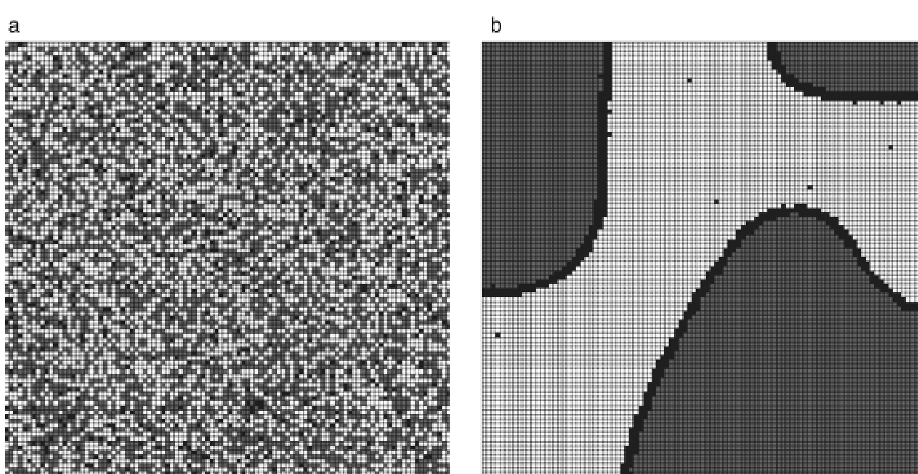


Figure 1. Persistent residential patterns of the standard Schelling model in the case of 95% density of agents: (a) $F = 0.2$, the residential pattern cannot be distinguished from a random one; (b) $F = 0.8$, the residential pattern is segregated.

characteristic of the model irrespective of neighborhood size and form. Laurie and Jaggi (2002, 2003) probed into whether Schelling model dynamics depend on the radius of a cell's neighborhood. They claimed that for any given *F* the number of homogeneous patches in the patterns decreases, with patch size increasing with the growth in neighborhood size. Portugali, Benenson, and Omer (1994) and Portugali and Benenson (1995) explore Schelling's claim that the symmetric behavior of the B- and G-agents is not at all necessary for segregation to emerge. They show that the model's residential pattern converges toward segregation even if agents of *one type only* react to neighborhood structure while agents of the second type are fully tolerant of the other's presence.

The ''bounded neighborhood'' version of the Schelling model (Schelling 1978, p. 155) considers residential space as divided into square ''blocks'' of cells. No matter where an agent is located, whether in a block's center or close to its boundary, this agent reacts to the fraction of friends within the block and chooses to leave its cell if this fraction is below *F*. The analytical and nonspatial investigations of this version of the model by Schelling (1978) and Clark (1991, 1993, 2002, 2006) demonstrate that, like the basic version of the model, if all agents demand a sufficiently high fraction of friends, then a block's population becomes homogeneous over time.

A spatial version of the bounded neighborhood model, in which agents account for the population structure of adjacent blocks, is investigated in depth in a series of recent simulations studies (Fossett and Waren 2005; Fossett 2006a, b; Clark and Fossett 2008). Modification of the model's assumptions reflects the residential behavior of the main ethnic groups—White, Black, and Hispanic—populating U.S. cities. As might be expected, system dynamics become more variable under these settings than in the basic Schelling model. Complexity going beyond the basic random-segregated dichotomy of residential patterns also is characteristic of agents who differed in two or more Boolean characteristics (Benenson 1998), as well as of those who exchange residences rather than search (as predicted by S5) for vacancies over all unoccupied cells (Pollicott and Weiss 2001; Zhang 2004).

To summarize, some studies of Schelling's model confirm the basic random-segregated dichotomy, while some modify it. We thus require a general theoretical insight into the variety of the patterns engendered by the Schelling model in order to reach the level of generalization necessary to encompass different results. A first step in this direction was recently taken by Vinkovic and Kirman (2006), who define and investigate a continuous, in space and in time, analog of the Schelling model. They propose distinguishing between two types of the relocation rules that result in two distinctive types of persistent patterns: ''solid'' and ''liquid.'' The rules that allow relocation to the *better location only* result in solid-like residential dynamics, that is, the system stalls when converging to a pattern in which none of the agents can improve their current state. The set of rules that allows relocation to *cells of the same utility* results in liquid-like dynamics, meaning that the pattern varies slowly and never stalls while preserving its integral characteristics, such as level of segregation, as agents migrate between equally satisfying locations. Vinkovic and

Kirman (2006) demonstrate that qualitatively different solid-like patterns are numerous, depending on the model settings (e.g., the number of empty cells). The persistent liquid-like patterns are simple and consist of two large clusters of agents of each type, irrespective of variations in the model details.

This article is a step toward systematic investigation of the traditional—discrete—Schelling model. Following Vinkovic and Kirman (2006), we search for sets of rules and rule parameters that result in solid-like and liquid-like dynamics. We examine the model as if it depends on the threshold fraction of friends F for two basic types of agent behavior: satisficing and maximizing (Simon 1982)–and for varying ratios of B and G population groups.

Our study confirms that the basic distinction between the rule sets that result in solid and liquid patterns, respectively, remain valid in the case of the discrete version of the Schelling model. The maximizing behavior of residential agents entails solid-like dynamics: the model's residential patterns are always segregated, their variety is high, and they are very sensitive to initial conditions and behavioral rule parameters. We present several representative examples of these dynamics while focusing on satisficing residential behavior, an option implemented by Schelling himself.

Agents' satisficing residential behavior entails liquid dynamics. With this kind of residential behavior, we discover additional persistent, nonrandom, and non-segregated patterns, observed when the numbers of the B- and G-groups differ. These new patterns consist of two parts, a homogeneous part, occupied exclusively by members of the majority and a ''mixed'' part, where the members of two groups coexist. The lower the fraction of the minority, the wider the interval of F-values allowing for mixed patterns. We next discuss the correspondence between the model results and real-world residential dynamics.

A detailed description of the model

Next, we consider, the Schelling model in discrete time and space, where a *city* is a *torus* grid of cells, populated by B- and G-agents who act in discrete time $t = 0, 1, 2, \ldots$ (we omit index t unless its meaning becomes ambiguous). At each time step, every agent decides whether and where to relocate. Agents are considered in a random order, which is established anew at each time step. We follow Schelling's condition S6 and assume that an agent observes the system's changes *immediately after* they occur. Formally, this view corresponds to asynchronous updating (Cornforth, Green, and Newth 2005).

Model rules and definitions

Consider a torus city of cells of size $N \times N$. Each cell can be occupied by only one residential agent. The fraction of empty cells in the city is $d > 0$. Agents can belong to one of two groups—B(lue) or G(reen). We assume the following:

- An agent located at cell h remains there, or relocates, in response to the fraction of friends (agents of the same group) within the neighborhood of h.

- An agent can relocate for reasons independent of the number of friends within its neighborhood (we refer to these as *random reasons*) with probability m.
- When relocating, an agent is able to consider and compare not more than w vacant locations; the awareness of agent a located at h about vacancy v does not depend on the distance between h and v.

Next we consider *Moore neighborhoods* of radius r; that is, a square of $(2r+1) \times (2r+1)$ size with h in the center (Moore 1970). For convenience, we exclude h itself from the neighborhood. For an agent a located at cell h, let us denote

- the neighborhood of h, excluding h itself, as $U(h)$;
- the fraction of a-type *friends* among the agents located within $U(h)$ as $f_a(h)$; and
- the minimal fraction of friends necessary for a's remaining within $U(h)$ as F.

Agent a's relocation is decided according to the following rules:
Step 1: *Decide whether to try to relocate*:

- Estimate $f_a(h)$ by dividing the number of agents of a-type within $U(h)$ by the overall number of agents there while ignoring h and the empty cells within $U(h)$.
- Generate a random number p, uniformly distributed on (0, 1).
- If $f_a(h) < F$ or $p < m$ decide to try to relocate; otherwise, decide to stay at h.

Step 2: *If the decision is to try to relocate, then search for a new location and decide whether to move there*:

- Construct set $V(a)$ of opportunities by randomly selecting w vacancies from all of the city's current vacancies.
- Estimate utility $u_a(v)$ of each $\in V(a)$ as $u_a(v) = \min(f_a(v), F)$; that is, follow the satisficing behavior rule and consider all vacancies with F or higher fractions of friends as having the same utility.
- Select the vacancy $v_{best} \in V(a)$ whose utility $u_a(v_{best})$ is the highest among all $v \in V(a)$. If there are several best vacancies, choose one at random.
- Move to v_{best} if
- the fraction of friends at the current location is low—that is, $u_a(h) < F$—and relocating would improve it; that is, $u_a(v_{best}) > u_a(h)$;
- the reason for relocating is other than an unfriendly neighborhood—that is, $u_a(h) \geq F$—whereas the new neighborhood is also friendly; that is, $u_a(v_{best}) \geq F$.
- Otherwise stay at h.

The sequence of model events is presented in Fig. 2. Note that an agent located at location a, for which $f_a < F$, relocates to a vacancy v only if $f_v > f_a$. An agent located at a, for which $f_a \geq F$, relocates only for ''random reasons,'' with probability m and, in accordance with the model rules, without improving its utility.

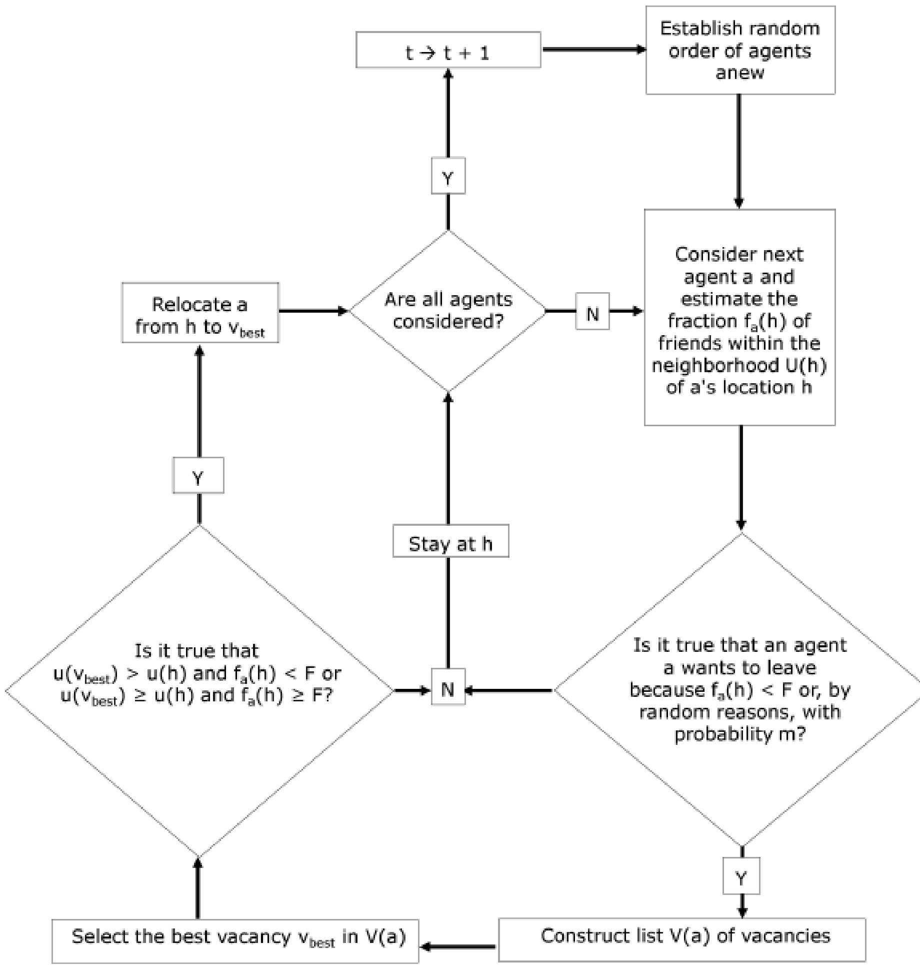


Figure 2. Flow chart of the simulation model.

We should emphasize that following Schelling (1978), model agents are ''satisficers'' (Simon 1982); that is, they consider two vacancies as equally good so far as both have a fraction of friends equal to or above *F*. We might thus expect liquid system dynamics (Vinkovic and Kirman 2006). Subsequently, we also discuss the ''maximizer's'' perspective when, for *a*, the utility $u_a(v)$ of location *v* equals the number of friends within $U(v)$.

We next specify the model parameters and describe the model's results.

Model settings and initial conditions

We consider a city represented by the 50×50 torus and investigate the model's dynamics for the following set of parameters:

- square (Moore) neighborhoods for $r = 1–4$;

- the fraction β of the minority B-group varying over [0, 0.5]; and
- the threshold intolerance F varying over [0, 1].

We assume that

- the probability of random relocation m is low but nonzero; accordingly, we employ $m = 0.01$ per time step;
- the majority of, but not all, cells in an urban space are occupied; accordingly, we employ the value $d = 0.02$; and
- an agent tests many vacancies per time step when considering relocation; accordingly, we employ $w = 30$.

Note that for a neighborhood of radius r, the number of friends can vary between 0 and $n(r) = (2r+1)(2r+1) - 1$. That is, for any given r, we can study the system's dynamics for any series F_k of $n(r)+1$ values of F, which satisfies the condition $(k-1)/n(r) < F_k \leq k/n(r)$ ($k = 0, 1, \ldots, n(r)$). To avoid rounding problems in what follows, we employ the series

$$F_k = k/n(r) - 1/(2n(r)), \qquad \text{where } k = 0, 1, \ldots, n(r), \tag{1}$$

$F = F_0$ provides the same results as $F = 0$ and corresponds to the case where agents are absolutely insensitive to neighbors. $F = F_1$ provides the same results as $F = 1/n(r)$ and is characteristic of agents who demand at least one friend in a fully occupied neighborhood. This progression continues until $F = F_{n(r)}$, which characterizes agents who demand a neighborhood occupied solely by friends.

The initial set of agents is established by initiating $M = \text{Round}\ ((1-d)N^2)$ agents and by assigning type B to the $M_B = \text{Round}\ (\beta(1-d)N^2)$ of them and type G to the remaining $M_G = M - M_B$ agents. The random pattern (as in Fig. 1a) is constructed by randomly locating all agents over the $N \times N$ city grid. The properties of the initial segregated pattern are similar to that in Fig. 1b, but the pattern itself is constructed by locating all B-agents on the $N \times N$ grid at 100% density, beginning from its left border, column after column, from the top to the bottom border within each column and, symmetrically, by locating G-agents beginning from the right border, column after column, from the bottom to the top border within each column.

Characterizing model patterns

Moran's I as an index of segregation

To characterize the model's spatial pattern, we define a spatial variable x_h: $x_h = 1$ if h is occupied by the B-agent and $x_h = 0$ otherwise; x_h is ignored if h is empty. Of the several ways to characterize a pattern's segregation level, we chose Moran's I index of spatial association (Getis and Ord 1992; Anselin 1995; Zhang and Linb 2007) applied to the binary data (Lee 2001; Griffith 2010).

Simulation modeling has difficulties revealing all possible dynamic regimes for a given set of parameters. One can easily miss stable equilibria with narrow domains of attraction, especially if other stable equilibria exist whose domains of attraction are wide. In attempting to avoid errors of this kind, we perform extensive numeric experiments with different sets of parameters before the investigation of the model.

Pattern randomness and segregation criteria

To recognize random-like patterns, we first conduct a permutation test for the 50×50 random city and obtain $(-0.02, 0.02)$ and $(-0.03, 0.03)$ as the 95% and 99% confidence intervals for Moran's I, respectively. Consequently, we consider the 50×50 pattern as nonrandom in the case of Moran's $I > 0.03$. In addition to Moran's I, we employ, when necessary, additional indices for characterizing non-random patterns.

In situations of liquid dynamics, the model patterns can vary. Nevertheless, a pattern's level of segregation can remain the same. To reflect this possibility, we call the model pattern characterized by Moran's $I = I_0$ "persistent" if the value of I_0 remains within the interval $(I_0 - 0.03,\ I_0 + 0.03)$ during at least 100,000 time steps.

The basic view of the Schelling model's dynamics

Our numeric experiments demonstrate that for random initial conditions, the agents' residential patterns either remain random infinitely for low F or converge to a segregated pattern for high F. In the latter case, the patterns' dynamics consist of two phases, characterized by the dynamics of the unhappy agents; that is, those located in neighborhoods where $f_a(h) < F$:

- In the first phase, the pattern contains many unhappy agents, the fraction of whom quickly drops to almost zero during some hundred time steps. During this phase, segregated patches appear and expand.
- In the second phase, the pattern changes slowly. The fraction of unhappy agents remains close to zero and therefore is uncharacteristic of the pattern. Some segregated patches continue to grow, while others contract, although the overall number of patches decreases. The segregated pattern, once it emerges is infinitely sustained.

For values of F at the boundary that separates the values generating random-like patterns from those generating segregated patterns, the first phase could be very long and varies in length between executions (Fig. 3); for an expanded discussion, see Durrett (1999).

Given F, β, and r, the duration of the first and second phases for the non-boundary values of F is determined by d and m. For the investigated values of $d = 0.02$ and $m = 0.01$, the duration of the first phase is at most some hundred time steps.

The long-run evolution of patterns that do not stall (i.e., are liquid-like) should be treated with care. Typical practice is to run the model for a million time steps (Vinkovic and Kirman 2006) in the hope that the pattern reaches a persistent state

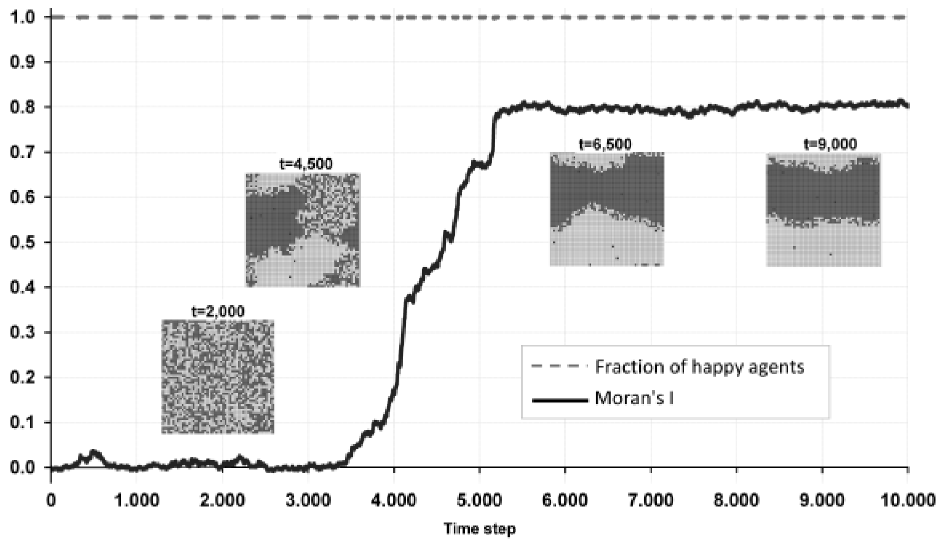


Figure 3. The model dynamics described by Moran's *I* and the fraction of happy agents for the boundary value of $F = 11/48$ ($r = 3$, $\beta = 0.5$).

by the end of this extremely long period. Ensuing results are based on the long run of $t = 500{,}000$ time steps.

Schelling's basic case: two groups of equal size ($\beta = 0.5$)

The case of $\beta = 0.5$ corresponds to Schelling's original formulation. Our goal is to reproduce the qualitative aspects of the model dynamics; that is, for low *F*, the residential distribution converges to a random pattern, whereas for high *F*, it converges to a segregated pattern irrespective of the initial conditions. To verify this prediction, we perform 30 executions/model runs for each value of *F* given by equation (1) for $r = 1$, 2, 3, and 4, as well as for two types of initial conditions, random and segregated.

Fig. 4a presents a set of typical emerging patterns and corresponding values of *I* for $r = 1$. To better characterize the model behavior as dependent on *F*, we denote the maximal *F*-value at which, for any initial distribution of agents, the pattern remains random for $t \to \infty$, by $F_{r,\,\mathrm{rand}}$. Similarly, let $F_{r,\,\mathrm{segr}}$ be the minimal *F*-value at which, for any initial distribution of agents, the pattern remains segregated for $t \to \infty$. According to our numeric experiments, at $t = 500{,}000$ for $r = 1$, 2, and 3, $F_{r,\,\mathrm{segr}} = F_{r,\,\mathrm{rand}} + 1/n(r)$; that is, the persistent model pattern does not depend on the initial conditions of these values of *r* (Table 1, rows 1–3). For $r = 4$, the situation is more complex (Table 1, row 4): for $F = 18/80$–$21/80$, the initially random pattern remains random, while the initially segregated pattern remains segregated up to $t = 500{,}000$. Based on Durrett (1999), we expect 500,000 time steps to be insufficient for pattern convergence in these cases; but investigation of these dynamics

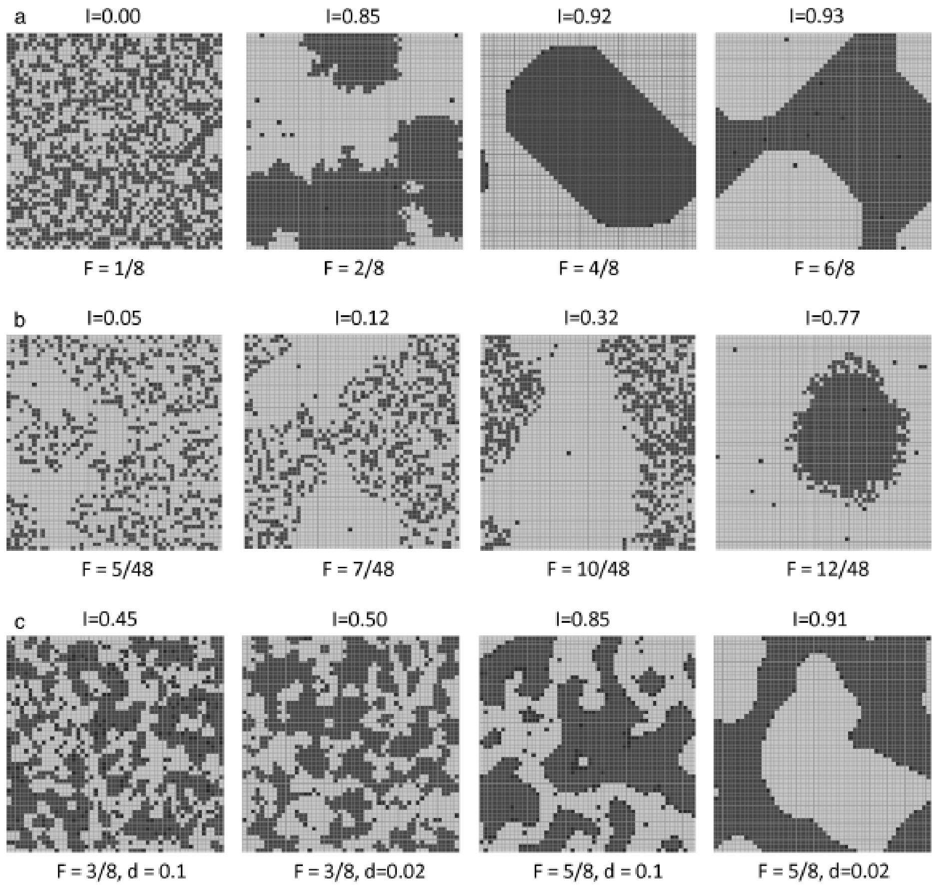


Figure 4. Persistent patterns for the representative sets of model parameters: (a) $r = 1$, $m = 0.01$, $d = 0.02$, $\beta = 0.5$, varying F; (b) $r = 3$, $m = 0.01$, $d = 0.02$, $\beta = 0.25$, varying F; (c) $r = 1$, $m = 0$, varying d, $\beta = 0.5$, varying F.

goes far beyond Schelling's view of the local determinants of residential dynamics and remains for future research. Note that for all r, the value of $F_{r,\,\text{segr}}$ is below the $F \sim 1/3$ given by Schelling.

Table 1 $F_{r,\,\text{rand}}$, $F_{r,\,\text{segr}}$ for $r = 1$–4, $\beta = 0.5$

r	$n(r)$	$F_{r,\,\text{rand}}$	$F_{r,\,\text{segr}}$
1	8	1/8 = 0.1250	2/8 = 0.2500
2	24	4/24 = 0.1667	5/24 = 0.2083
3	48	10/48 = 0.2083	11/48 = 0.2292
4	80	17/80 = 0.2125*	21/80 = 0.2625*

Note: The city is a 50 × 50 torus. *For $F = 18/80$, 19/80, and 20/80, the initially random pattern remains random at $t = 500{,}000$, while the initially segregated pattern remains segregated at $t = 500{,}000$.

We now extend the standard framework and investigate the dynamics of Schelling's model in the case where B-agents are the minority.

The minority group's patterns ($\beta < 0.5$)

The new mixed patterns

Study of the Schelling model's dynamics for the case where the B-agents are the minority (i.e., $\beta < 0.5$) reveals major new patterns, yet to be ascribed to this model, that are neither random nor completely segregated. We henceforth call these patterns *mixed*. In mixed patterns, a part of the area is exclusively occupied by the members of the majority, while the remaining area is occupied by agents of both types (Fig. 4b).

Mixed patterns: characterization

The patterns in Fig. 4b clearly show that we need to characterize the segregated as well as aggregated parts prior to investigating mixed patterns. Moran's *I* calculated for the entire city is insufficient for this purpose.

To characterize a mixed pattern, we divide it into four regions (Fig. 5). First, let $G^{\wedge}$ be the part of the city occupied by the majority group G exclusively. We distinguish between internal part $iG^{\wedge}$ and the boundary $nG^{\wedge}$ of $G^{\wedge}$. A cell *h* belongs to $iG^{\wedge}$ if *h* and all cells of the $U(h)$ are either occupied by G-agents or are empty. Cells that belong to the neighborhood of some cell from $iG^{\wedge}$, but not to $iG^{\wedge}$ itself, comprise $nG^{\wedge}$.

Second, let G^{+} complement the $G^{\wedge}$ part of the city. In the case of a segregated city, G^{+} is a homogeneous part occupied by the minority; otherwise, G^{+} represents the heterogeneous part of the city. We define the internal part iG^{+} of G^{+} as all cells *h* whose neighborhood $U(h)$ does not intersect $G^{\wedge}$, and the boundary nG^{+} as the

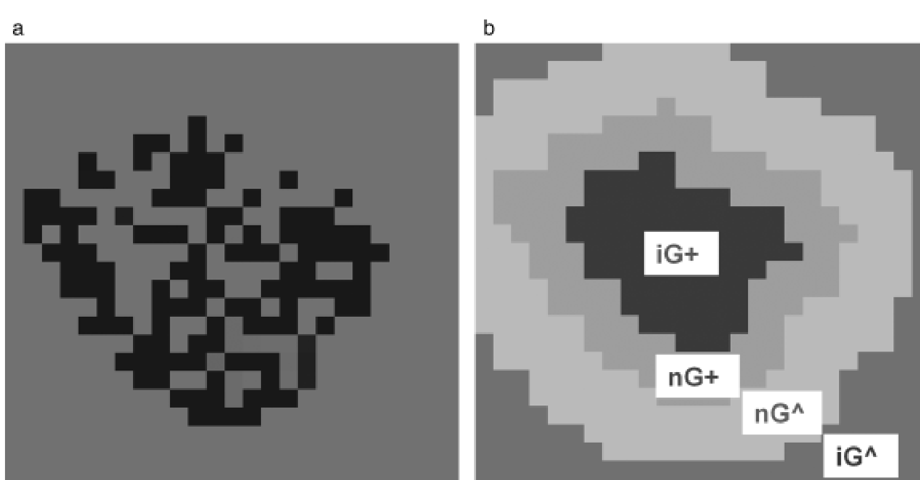


Figure 5. (a) A mixed pattern; (b) partition of the mixed pattern into four parts—$iG^{\wedge}$, $nG^{\wedge}$, nG^{+}, and iG^{+}.

remaining G^+ cells. With an increase in r, $iG^{\wedge}$ and iG^+ shrink, whereas $nG^{\wedge}$ and nG^+ expand.

In addition to Moran's I calculated for the entire city, we characterize the mixed patterns by three measures applied to the mixed part G^+:

- β^+: the fraction of minority agents within G^+;
- I^+: the value of Moran's I calculated for the internal part $iG^+(\beta)$ of the heterogeneous area; and
- $S(G^+)$: the area of G^+ as a fraction of the entire city area.

Study of the mixed patterns

The hypothesis formulated on the basis of Fig. 4 is as follows: for $\beta < 0.5$, three types of persistent patterns—random, mixed, and fully segregated—substitute for one another as F grows.

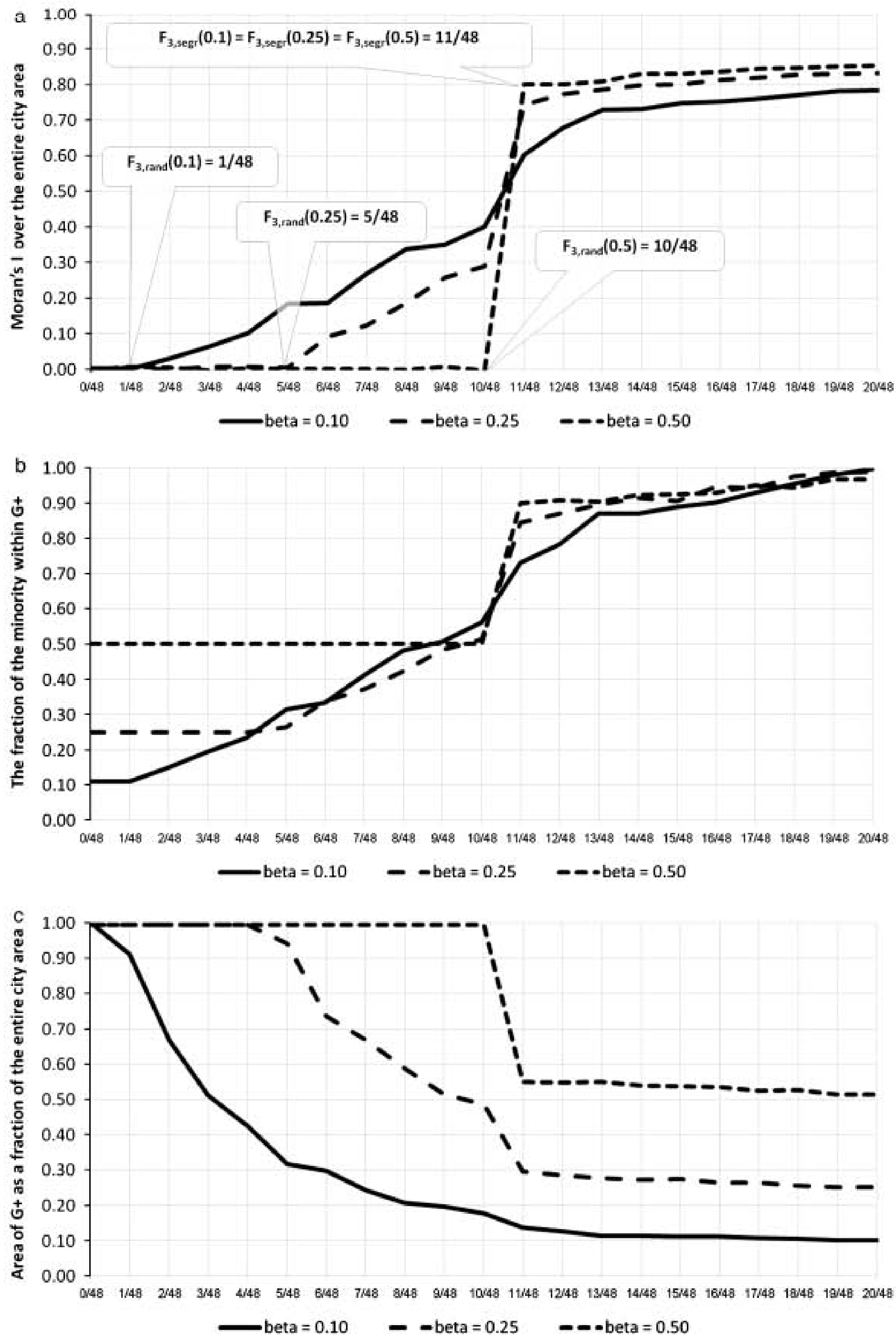
To test this hypothesis, we denote as $F_{r,\,\mathrm{rand}}(\beta)$ the maximal F-value at which, given β, the persistent pattern converges to random for any initial distribution of agents when $t \to \infty$. We denote as $F_{r,\,\mathrm{segr}}(\beta)$ the minimal F-value at which, given β, the pattern *converges to segregated* for any initial distribution of agents when $t \to \infty$. To avoid complications related to the difference between $F_{r,\,\mathrm{rand}}(\beta)$ and $F_{r,\,\mathrm{segr}}(\beta)$, obtained for $\beta = 0.5$ when $r = 4$, we limit ourselves to cases of $r = 1$–3.

By definition, the city is random for $F \in [0, F_{r,\,\mathrm{rand}}(\beta)]$, and the city is segregated for $F \in [F_{r,\,\mathrm{segr}}(\beta), 1]$. As demonstrated in the section "Schelling's basic case," for $r = 1$–3 and $\beta = 0.5$, $F_{r,\,\mathrm{rand}}(0.5)$ and $F_{r,\,\mathrm{segr}}(0.5)$ are sequential values of F_k, as given by formula (1); that is, $F_{r,\,\mathrm{segr}}(0.5) = F_{r,\,\mathrm{rand}}(0.5) + 1/n(r)$. We hypothesize that for $\beta < 0.5$ the interval $[F_{r,\,\mathrm{rand}}(\beta), F_{r,\,\mathrm{segr}}(\beta) - 1/n(r)]$ is nonzero; its width increases with a decrease in β. For F, the persistent pattern is mixed within this interval.

To confirm this hypothesis, we perform a set of simulation experiments for the values of $r = 1$–3, with β varying from 0.05 to 0.45, by increments of 0.05. Typical dependencies of Moran's I for the entire city, β^+, and $S(G^+)$ on F for $r = 3$ and for $\beta = 0.1$, 0.25 and 0.5, are shown in Fig. 6.

Analysis of the full set of simulation results implies the following conclusions:

- The value of $F_{r,\,\mathrm{segr}}(\beta)$ does not depend on β; that is, $F_{r,\,\mathrm{segr}}(\beta) = F_{r,\,\mathrm{segr}}$ (Fig. 6a–c).
- The lower β, the lower $F_{r,\,\mathrm{rand}}(\beta)$ (Fig. 6a).
- The pattern of the $iG^+(\beta)$ is random for $F \in [0, F_{r,\,\mathrm{segr}} - 1/n(r)]$ and populated exclusively by the minority for $F \in [F_{r,\,\mathrm{segr}}, 1]$ (Fig. 6a).
- The density β^+ of the B-agents in $G^+(\beta)$ is always higher than β, grows with an increase in F within $[F_{r,\,\mathrm{rand}}(\beta), F_{r,\,\mathrm{segr}} - 1/n(r)]$, and always bypasses $\beta^+ = 0.5$ for $F = F_{r,\,\mathrm{segr}} - 1/n(r)$ (Fig. 6b).
- For values of F close to zero, $G^+(\beta)$ covers the city's entire area. With the growth of F within the interval $[F_{r,\,\mathrm{rand}}(\beta), F_{r,\,\mathrm{segr}} - 1/n(r)]$, the area of $G^+(\beta)$ decreases, stabilizing at β for $F \geq F_{r,\,\mathrm{segr}}$ (Fig. 6c).[3]


a
F3,segr(0.1) = F3,segr(0.25) = F3,segr(0.5) = 11/48
F3,rand(0.1) = 1/48
F3,rand(0.25) = 5/48
F3,rand(0.5) = 10/48
Moran's I over the entire city area
0/48 1/48 2/48 3/48 4/48 5/48 6/48 7/48 8/48 9/48 10/48 11/48 12/48 13/48 14/48 15/48 16/48 17/48 18/48 19/48 20/48
beta = 0.10
beta = 0.25
beta = 0.50
b
The fraction of the minority within G+
c
Area of G+ as a fraction of the entire city area

The model's robustness

Let us consider the importance of the three qualitative assumptions made:

- a low but nonzero fraction of empty cells *d*;
- a nonzero probability of spontaneous migration *m*; and
- satisficing behavior by agents.

The values of d and m: The values we employ, $d = 0.02$ and $m = 0.01$, are chosen to reflect a real city, where empty locations (although perhaps not many) are always available for occupation. Additionally, abundant reasons exist for residential migration beyond reactions to neighbors.

In the case of the 50 × 50 city that we use for simulation experiments, $d = 0.02$ results in 50 empty locations. We also investigate the model's behavior for a lower *d*; that is, for fewer empty locations in the city. Simulation results do not change until the fraction of empty locations falls below $d = 0.01$. For a few empty locations and for some of the random initial patterns, the model stalls because all vacancies are of low utility, meaning that no agents want to relocate there. As should be expected, the fraction of the initial patterns that stall grows with a decrease in *d*.

The value of $m > 0$ also prevents the model's pattern from stalling in the solid-like state, where no agents can improve their utility. We test the sensitivity of the model's persistent patterns to variations of *m* for $m \in (0.01, 0.05)$ and find that they do not fundamentally change, whereas an increase in *m* accelerates and a decrease in *m* decelerates convergence to a persistent pattern. In the case of $m = 0$, the model generates a variety of solid-like patterns, as shown in Fig. 4c.

The simulations clearly demonstrate that in the solid-like state the structure of the stalled pattern is very sensitive to the model's parameters and rule details, such as whether the fraction of friends is calculated by dividing by the size of an entire neighborhood, as in Vinkovic and Kirman (2006), or by the number of occupied cells in the neighborhood, as in our simulation.

Agents' satisficing behavior: According to the behavioral rules (See ''Model rules and definitions''), the model's agents are satisficers (Simon 1982); that is, they do not distinguish among potential locations provided that the number of friends is above the threshold. To investigate the importance of this assumption, we replace the satisficing choice rule with the maximization rule: the more friends the better. Formally, maximizing agent a estimates the utility of vacancy *h* as $u(h) = f_a(h)$. This

◀

Figure 6. Characteristics of persistent patterns for $r = 3$ depending on minority fraction β and F, $\beta = 0.05$–0.5 with steps of 0.05, $F = 0/48$–$20/48$ with steps of 1/48, and random initial conditions: (a) Moran's *I* for the entire city; (b) the fraction β^+ of minority agents within $G^+(\beta)$; (c) $S(G^+(\beta))$ as a fraction of the entire city area. $F_{r,\,rand}(\beta)$ and $F_{r,\,segr}(\beta)$ are marked on (a) only but are applicable to (a)–(c). Note that $G^+(\beta)$ is the union of the iG^+ and nG^+ domains as shown in Fig. 5.

assumption's consequence is as expected, *Maximizers always segregate*. Irrespective of F, the pattern always converges to complete segregation.

Conclusions and discussion

Our results essentially extend Schelling's (1978, pp. 147–48) observations. Given a neighborhood's radius r and the fraction of minority β, the model's pattern converges to one of three—and not two, as Schelling claims—persistent states. Each persistent state is defined by F, with the $[0, 1]$ interval of F variability divided into three parts: for $F \in [0, F_{r,\text{rand}}(\beta)]$, the persistent residential pattern is random; for $F \in (F_{r,\text{rand}}(\beta),\ F_{r,\text{segr}})$, it is mixed; and for $F \in [F_{r,\text{segr}}, 1]$, it is segregated.

The mixed urban pattern, characteristic of $F \in (F_{r,\text{rand}}(\beta),\ F_{r,\text{segr}})$, consists of a homogeneous part, occupied by the majority, and a heterogeneous part, randomly occupied by agents of both types. The density of the minority in the mixed part is always greater than β and grows with F.

After investigating the model's formal properties, the principal question becomes: how can we relate the model to real-world residential dynamics? Three persistent residential patterns are obtained with the model when four conditions are met:

(1) The fraction d of empty locations in a city is sufficiently high. The value employed in the simulations is $d = 0.02$; an additional investigation shows that liquid dynamics are sustained until d remains above ~ 0.01. For lower fractions of empty locations, the system stalls in a state where none of the agents who are unhappy with their neighborhood can reside elsewhere.

(2) The probability m that an agent would reside in a neighborhood irrespective of the neighborhood structure is nonzero. The value employed in our simulations is $m = 0.01$; that is, 1% of the agents decide to move at each time step despite their residence in a friendly neighborhood. For $m = 0$, the residential pattern stalls, similar to what happens when the number of empty locations is insufficient for migration.

(3) The radius r of the neighborhood is three or lower. For $r > 3$, model dynamics may depend on initial conditions: for the same set of parameters, initially random patterns remain random, while initially segregated patterns remain segregated. This phenomenon demands further investigation. Until then, we can qualitatively conclude that the model results are robust in cases when agents react to neighbors located in sufficiently close proximity to their actual or potential location.

(4) Most importantly, the behavior of the model agents must be satisficing. Namely, agents should not distinguish between locations so long as the fraction of friends around those locations is $\geq F$. Alternatively, if we assume that agents always relocate to locations within the neighborhood having the maximum possible fraction of friends, the model dynamics become solid-like, residential patterns become segregated for any F, and a great variety of stalled and segregated patterns are obtained, depending on the model parameters and initial conditions.

We thus conclude that the conditions necessary for solid model dynamics are unrealistic. Socially meaningful extensions and interpretations of the Schelling model should consider a population with sufficiently high fractions of satisficing agents, whose behavior provides a sufficient degree of the indeterminism necessary for liquid-like system dynamics.

Our recent research about ethnic residential dynamics in Israeli cities, conducted at the resolution level of the separate family and building and employing the individual Israeli population census for 1995 (Benenson, Omer, and Hatna 2002; Benenson, Hatna, and Or 2009), supports the view of mixed residential patterns. In Israeli cities with mixed Jewish and Arab populations of similar socioeconomic status—Yaffo in Tel Aviv, Ramle, and Lod—the Arab minority is concentrated within some neighborhoods; the remaining areas of these cities are populated almost exclusively by the Jewish majority. At a closer glance, however, the population of the Arab neighborhoods is mixed, and the fraction of the Jewish population in these mixtures varies between 30% and 60%. In other words, the real-world patterns correspond to those obtained in the model for the case of relatively low *F*. The latter can be interpreted as indicating the relatively high tolerance of Jewish and Arab residential agents for one another. We consider this correspondence as encouraging and plan to repeat the investigation in additional cities and in greater detail after the data collected during the individual-based Israeli population census for 2008 becomes officially available, toward the end of 2011. The merger of data from two censuses (1995 and 2008) also will enable construction of individual migration records, facilitating investigation of the relationship between individual residential choices and group residential patterns during the respective period (1995–2008).

Notes

1 The number of neighbors in the neighborhood is finite, so it does not matter whether "less than" or "less than or equal" is employed in the definition; we employ Schelling's original use of "less" versus "greater than or equal."

2 http://www.econ.iastate.edu/tesfatsi/demos/schelling/schellhp.htm (accessed April 4, 2011) and R. Koenig's application at http://www.entwurfsforschung.de/RaumProzesse/Segregation.htm (accessed April 4, 2011) can be starting points. More complex situations involving agents differing in several characteristics can be investigated with the help of M. Fosset's application; see http://sociweb.tamu.edu/vlabresi/vlab.htm (accessed April 4, 2011).

3 Some of these simulation results can be confirmed analytically; see Benenson and Hatna (2009).

References

Anselin, L. (1995). "Local Indicators of Spatial Association—LISA." *Geographical Analysis* 27(2), 93–115.

Benenson, I. (1998). ''Multi-Agent Simulations of Residential Dynamics in the City.'' *Computers, Environment and Urban Systems* 22, 25–42.

Benenson, I., and E. Hatna. (2009). ''The Third State of the Schelling Model of Residential Dynamics.'' Available at http://arxiv.org/abs/0910.2414 (accessed April 19, 2011).

Benenson, I., E. Hatna, and E. Or. (2009). ''From Schelling to Spatially Explicit Modeling of Urban Ethnic and Economic Residential Dynamics.'' *Sociological Methods Research* 2009; 37(4), 463–97.

Benenson, I., I. Omer, and E. Hatna. (2002). ''Entity-based Modeling of Urban Residential Dynamics—The Case of Yaffo, Tel-Aviv.'' *Environment and Planning B* 29, 491–512.

Clark, W. A. V. (1991). ''Residential Preferences and Neighborhood Racial Segregation: A Test of the Schelling Segregation Model.'' *Demography* 28(1), 1–18.

Clark, W. A. V. (1993). ''Search and Choice in Urban Housing Markets.'' In *Behavior and Environment: Psychological and Geographical Approaches*, 298–316, edited by T. Gerling and R. G. Golledge. Amsterdam: Elsevier/North Holland.

Clark, W. A. V. (2002). ''Ethnic Preferences and Ethnic Perceptions in Multi-Ethnic Settings.'' *Urban Geography* 23, 237–56.

Clark, W. A. V. (2006). ''Ethnic Preferences and Residential Segregation: A Commentary on Outcomes from Agent-Based Modeling.'' *Journal of Mathematical Sociology* 30(3), 319–26.

Clark, W. A. V., and M. Fossett. (2008). ''Understanding the Social Context of the Schelling Segregation Model.'' *Proceedings of the National Academy of Sciences* 105(11), 4109–14.

Cornforth, D., D. G. Green, and D. Newth. (2005). ''Ordered Asynchronous Processes in Multi-Agent Systems.'' *Physica D* 204, 70–82.

Durrett, R. (1999). ''Stochastic Spatial Models.'' *SIAM Review* 41(4), 677–718.

Flache, A., and R. Hegselmann. (2001). ''Do Irregular Grids Make a Difference? Relaxing the Spatial Regularity Assumption in Cellular Models of Social Dynamics.'' *Journal of Artificial Societies and Social Simulation* 4(4). Available at http://jasss.soc.surrey.ac.uk/4/4/6.html (accessed April 9, 2011).

Fossett, M. A. (2006a). ''Ethnic Preferences, Social Distance Dynamics, and Residential Segregation: Theoretical Explorations Using Simulation Analysis.'' *Mathematical Sociology* 30(3–4), 185–273.

Fossett, M. A. (2006b). ''Including Preference and Social Distance Dynamics in Multi-Factor Theories of Segregation.'' *Mathematical Sociology* 30(3–4), 289–98.

Fossett, M. A., and W. Waren. (2005). ''Overlooked Implications of Ethnic Preferences for Residential Segregation in Agent-based Models.'' *Urban Studies* 42(11), 1893–917.

Getis, A., and J. Ord. (1992). ''The Analysis of Spatial Association by Use of Distance Statistics.'' *Geographical Analysis* 24, 189–206.

Griffith, D. (2010). ''The Moran Coefficient for Non-Normal Data.'' *Journal of Statistical Planning and Inference* 140, 2980–90.

Laurie, A. J., and N. K. Jaggi. (2002). ''Physics and Sociology: Neighbourhood Racial Segregation.'' *Solid State Physics* 45, 183–84.

Laurie, A. J., and N. K. Jaggi. (2003). ''Role of Vision in Neighbourhood Racial Segregation: A Variant of the Schelling Segregation Model.'' *Urban Studies* 40(13), 2687–704.

Lee, S.-I. (2001). ''Developing a Bivariate Spatial Association Measure: An Integration of Pearson's r and Moran's I.'' *Journal of Geographical Systems* 3, 369–85.

Moore, E. F. (1970). ''Machine Models of Self-Reproduction.'' In *Essays on Cellular Automata*, 187–203, edited by A. W. Burks. Urbana: University of Illinois Press.
Pollicott, M., and H. Weiss. (2001). ''The Dynamics of Schelling-Type Segregation Models and a Nonlinear Graph Laplacian Variational Problem.'' *Advances in Applied Mathematics* 27, 17–40.
Portugali, J., and I. Benenson. (1995). ''Artificial Planning Experience by Means of a Heuristic Cell-Space Model: Simulating International Migration in the Urban Process.'' *Environment and Planning B* 27, 1647–65.
Portugali, J., I. Benenson, and I. Omer. (1994). ''Socio-Spatial Residential Dynamics: Stability and Instability within a Self-Organized City.'' *Geographical Analysis* 26(4), 321–40.
Schelling, T. C. (1971). ''Dynamic Models of Segregation.'' *Journal of Mathematical Sociology* 1, 143–86.
Schelling, T. C. (1978). *Micromotives and Macrobehavior*. New York: Norton.
Simon, H. A. (1982). *Models of Bounded Rationality*. Cambridge: MIT Press.
Vinkovic, D., and A. Kirman. (2006). ''A Physical Analogue of the Schelling Model.'' *PNAS* 103(51), 19261–65.
Zhang, J. (2004). ''Residential Segregation in an All-Integrationist World.'' *Journal of Economic Behavior and Organization* 54, 533–50.
Zhang, T., and G. Linb. (2007). ''A Decomposition of Moran's I for Clustering Detection.'' *Computational Statistics and Data Analysis* 51, 6123–37.